

ON THE STABILITY OF MINIMAL CONES IN WARPED PRODUCTS

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ABSTRACT. In a seminal paper published in 1968, J. Simons proved that, for $n \leq 5$, the Euclidean (minimal) cone CM , built on a closed, oriented, minimal and non totally geodesic hypersurface M^n of \mathbb{S}^{n+1} is unstable. In this paper, we extend Simons' analysis to *warped* (minimal) cones built over a closed, oriented, minimal hypersurface of a leaf of suitable warped product spaces. Then, we apply our general results to the particular case of the warped product model of the Euclidean sphere, and establish the instability of CM , whenever $2 \leq n \leq 14$ and M^n is a closed, oriented, minimal and non totally geodesic hypersurface of \mathbb{S}^{n+1} .

1. INTRODUCTION

In 1968, J. Simons (cf. [10]) generalized a theorem of F. J. Almgren, Jr. (cf. [1]), showing that, for $n \leq 5$, the Euclidean cone built over any closed, minimal and non totally geodesic hypersurface of \mathbb{S}^{n+1} is a minimal unstable hypersurface of \mathbb{R}^{n+2} .

If M^n is a hypersurface of \mathbb{S}^{n+1} , the Euclidean cone CM over M is given by the immersion $\Phi : M^n \times (0, 1] \rightarrow \mathbb{R}^{n+2}$, that sends (p, t) to tp . For $0 < \epsilon < 1$, the ϵ -truncated cone $C_\epsilon M$ over M is the restriction of Φ to $M \times [\epsilon, 1]$. In [10], it is shown that, if M^n is a closed minimal hypersurface of \mathbb{S}^{n+1} , then $CM \setminus \{0\}$ is a minimal hypersurface of \mathbb{R}^{n+2} ; moreover, $C_\epsilon M$ is compact and such that $\partial(C_\epsilon M) = M \cup M_\epsilon$, where $M_\epsilon = \{\epsilon p; p \in M\}$.

In [4], the second author extended this notion in the following way: let \overline{M}_c^{n+2} be a Riemannian manifold whose sectional curvature is constant and equal to c . Assume that \overline{M} admits a closed conformal vector field $\xi \in \mathfrak{X}(\overline{M})$, with conformal factor ψ_ξ . If $\xi \neq 0$ on \overline{M} , it is well known that the distribution $\{\xi\}^\perp$ is integrable, with leaves totally umbilical in \overline{M} . Let Ξ^{n+1} be such a leaf and $\varphi : M^n \rightarrow \Xi^{n+1}$ be a closed hypersurface Ξ . If Ψ denotes the flow of $\xi/\|\xi\|$, the compactness of M guarantees the existence of $\epsilon > 0$ such that Ψ is well defined in $[-\epsilon, 0] \times \varphi(M)$, and the mapping $\Phi : M^n \times [-\epsilon, 0] \rightarrow \overline{M}_c^{n+2}$ that sends (p, t) to $\Psi(t, \varphi(p))$ is also an immersion. By furnishing $M^n \times [-\epsilon, 0]$ with the induced metric, we turn Φ into an isometric immersion such that $\Phi|_{M^n \times \{0\}} = \varphi$; the Riemannian manifold $M^n \times [-\epsilon, 0]$, is the ϵ -truncated cone $C_\epsilon M$ over M , in the direction of ξ .

When \overline{M}_c^{n+2} is a warped product $I \times_f F^{n+1}$, with $I \subset \mathbb{R}$, it is well known that $\xi = (f \circ \pi_I)\partial_t$ ($\pi_I : \overline{M} \rightarrow I$ being the canonical projection) is a closed conformal vector field on \overline{M} , with conformal factor $\psi_\xi = f' \circ \pi_I$. If we ask that $f(0) = 1$, then $\Xi = \{0\} \times F$, furnished with the induced metric, is a leaf of the distribution $\{\xi\}^\perp$ and is isometric to F . Hence, one can identify an isometric immersion $\varphi : M^n \rightarrow F^{n+1}$ with the isometric immersion $\tilde{\varphi}(p) = (0, \varphi(p))$, from M^n into $\Xi = \{0\} \times F$. The flux

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of $\xi/\|\xi\|$ is given by $\Psi(t, (x, p)) = (t + x, p)$, so that $\Phi(p, t) = \Psi(t, \tilde{\varphi}(p)) = (t, \varphi(p))$ and $C_\epsilon M$ can be identified to the immersion $\Phi : M^n \times [-\epsilon, 0] \rightarrow I \times_f F^{n+1}$, that sends (p, t) to $(t, \varphi(p))$.

Our goal in this paper is to analyse the stability of $C_\epsilon M$ when M^n is a closed minimal hypersurface of F^{n+1} . In doing so, we extend a result of Simons (cf. [10]), proving the following assertion (cf. Theorem 3.3 and equations (25) and (26)).

Theorem. *In the above setting, $C_\epsilon M$ is unstable if, and only if, $\lambda_1 + \delta_1 < 0$, where λ_1 and δ_1 are, respectively, the first eigenvalues of the linear differential operators $\mathcal{L}_1 : C^\infty(M) \rightarrow C^\infty(M)$ and $\mathcal{L}_2 : C_0^\infty[-\epsilon, 0] \rightarrow C^\infty[-\epsilon, 0]$, given by*

$$\mathcal{L}_1(g) = -\Delta g - \|A\|^2 g \quad \text{and} \quad \mathcal{L}_2(h) = -f^2 h'' - n f f' h' - c(n+1) f^2 h,$$

for $g \in C^\infty(M)$ and $h \in C_0^\infty[-\epsilon, 0]$.

Here, as in [10], $C_0^\infty[-\epsilon, 0] = \{h \in C^\infty[-\epsilon, 0]; h(-\epsilon) = h(0) = 0\}$.

Then, we specialize our discussion to the case of *spherical cones*. More precisely, we let \mathbb{S}^{n+1} be the equator of \mathbb{S}^{n+2} with respect to the North pole $N = (0, 1)$ of \mathbb{S}^{n+2} , and identify each $x \in \mathbb{S}^{n+1}$ with $(x, 0) \in \mathbb{S}^{n+2}$; if we let $\overline{M} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{S}^{n+1}$, then the mapping $(t, x) \mapsto (\cos t)x + (\sin t)N$ defines an isometry between \overline{M} and $\mathbb{S}^{n+2} \setminus \{\pm N\}$. The ϵ -truncated cone $C_\epsilon M$ in \mathbb{S}^{n+2} , built over a closed minimal hypersurface M^n of \mathbb{S}^{n+1} is given by the immersion $\Phi : M^n \times [-\epsilon, 0] \rightarrow \mathbb{S}^{n+2}$ that maps (x, t) to $(\cos t)x + (\sin t)N$. In this setting, we finish the paper by proving the following result (cf. Theorem 4.2).

Theorem. *Let M^n be a closed, oriented minimal hypersurface of \mathbb{S}^{n+1} . If $2 \leq n \leq 14$ and M^n is not totally geodesic, then CM is a minimal unstable hypersurface of \mathbb{S}^{n+2} .*

2. ON FOLIATIONS GENERATED BY CLOSED CONFORMAL VECTOR FIELDS

In what follows, \overline{M}_c^{n+k+1} is an $(n+k+1)$ -dimensional Riemannian manifold, whose sectional curvature is constant and equal to c . We assume that \overline{M} is furnished with a nontrivial closed conformal vector field ξ , i.e., $\xi \in \mathfrak{X}(M) \setminus \{0\}$ is such that $\nabla_X \xi = \psi_\xi X$, for all $X \in \mathfrak{X}(M)$, where $\psi_\xi : \overline{M} \rightarrow \mathbb{R}$ is a smooth function, said to be the *conformal factor* of ξ , and ∇ denotes the Levi-Civita connection of \overline{M} .

From now on, the condition that $\xi \neq 0$ on \overline{M} will be in force. It is immediate to check (cf. [4]) that the distribution $\{\xi^\perp\}$ is integrable, with leaves totally umbilical in \overline{M} . Let Ξ^{n+k} be a leaf of such distribution, M^n be a closed, n -dimensional Riemannian manifold and $\varphi : M^n \rightarrow \Xi^{n+k}$ be an isometric immersion. If we let $\Psi(t, \cdot)$ denote the flow of the vector field $\xi/\|\xi\|$, the compactness of M assures that we can choose $\epsilon > 0$ such that the map

$$(1) \quad \begin{array}{ccc} \Phi : & M^n \times [-\epsilon, 0] & \longrightarrow \overline{M}^{n+k+1} \\ & (p, t) & \longmapsto \Psi(t, \varphi(p)) \end{array}$$

is an immersion. The ϵ -truncated cone over M , in the direction of ξ , which will be henceforth denoted by $C_\epsilon M$, is the manifold with boundary $M^n \times [-\epsilon, 0]$, furnished with the metric induced by Φ . We observe that $C_\epsilon M$ is a compact, immersed submanifold of \overline{M}_c^{n+k+1} , such that $\partial(C_\epsilon M) = M \cup M_\epsilon$, where $M_\epsilon = \{\Psi(-\epsilon, \varphi(p)); p \in M\}$. At times, if there is no danger of confusion, we shall refer simply to the ϵ -truncated cone $C_\epsilon M$.

From now on, we will frequently refer to the smooth function $\lambda : M \times [-\epsilon, 0] \rightarrow \mathbb{R}$, given by

$$(2) \quad \lambda(q, t) = \exp \left(\int_0^t \frac{\psi_\xi}{\|\xi\|} (\Psi(s, \varphi(q))) ds \right).$$

The following result relates the second fundamental form of $C_\epsilon M$ at distinct points along the same generatrix of the cone.

Proposition 2.1. *Let A_q^η denote the shape operator of φ at q , in the direction of the unit vector η , normal to $T_q M$ in $T_q \Xi$. Let N denote the parallel transport of η along the integral curve of $\xi/\|\xi\|$ that passes through q . If $A_{(q,t)}^N$ denotes the shape operator of Φ at the point (q, t) , in the direction of $N_{(q,t)}$, then*

$$\|A_{(q,t)}^N\| = \frac{1}{\lambda(q, t)} \|A_q^\eta\|.$$

Proof. Fix a point $p \in M$ and, in a neighborhood $\Omega \subset M$ of p , an orthonormal set $\{e_1, \dots, e_n, \eta\}$ of vector fields, with e_1, \dots, e_n tangent to M^n and η normal to M^n in Ξ^{n+k} . Further, ask that $A_p^\eta(e_i) = \lambda_i e_i(p)$, for $1 \leq i \leq n$. Let E_1, \dots, E_n, N be the vector fields on $\Phi(\Omega \times (-\epsilon, 0])$, respectively obtained from e_1, \dots, e_n and η by parallel transport along the integral curves of $\frac{\xi}{\|\xi\|}$ that intersect Ω .

If we let \bar{R} denote the curvature operator of \bar{M} and use the fact that \bar{M} has constant sectional curvature, such a parallelism gives

$$\begin{aligned} \frac{d}{dt} \langle \bar{\nabla}_{E_i} N, E_k \rangle &= \langle \bar{\nabla}_{\frac{\xi}{\|\xi\|}} \bar{\nabla}_{E_i} N, E_k \rangle \\ &= \frac{1}{\|\xi\|} [\langle \bar{R}(\xi, E_i) N, E_k \rangle + \langle \bar{\nabla}_{E_i} \bar{\nabla}_\xi N, E_k \rangle + \langle \bar{\nabla}_{[\xi, E_i]} N, E_k \rangle] \\ (3) \quad &= \frac{1}{\|\xi\|} [\langle \bar{R}(\xi, E_i) N, E_k \rangle - \langle \bar{\nabla}_{\bar{\nabla}_{E_i} \xi} N, E_k \rangle] \\ &= -\frac{\psi_\xi}{\|\xi\|} \langle \bar{\nabla}_{E_i} N, E_k \rangle. \end{aligned}$$

Moreover, if D denotes the Levi-Civita connection of Ξ^{n+k} , then

$$(4) \quad \langle \bar{\nabla}_{E_i} N, E_k \rangle_{(p,0)} = \langle D_{e_i} \eta, e_k \rangle_p = -\langle A^\eta(e_i), e_k \rangle_p = -\lambda_i \delta_{ik}.$$

Equations (3) and (4) compose a Cauchy problem, whose solution is

$$\langle \bar{\nabla}_{E_i} N, E_i \rangle_{(p,t)} = -\lambda_i \exp \left(-\int_0^t \frac{\psi_\xi}{\|\xi\|} (\varphi(p), s) ds \right) = \frac{-\lambda_i}{\lambda(p, t)}$$

and, for $k \neq i$,

$$\langle \bar{\nabla}_{E_i} N, E_k \rangle_{(p,t)} = 0,$$

for all $t \in (-\epsilon, 0]$.

Since $\langle \bar{\nabla}_{E_i} N, \xi \rangle = -\langle N, \bar{\nabla}_{E_i} \xi \rangle = -\psi_\xi \langle N, E_i \rangle = 0$, it follows from the previous formulae that, at the point (p, t) ,

$$A^N(E_i) = -(\bar{\nabla}_{E_i} N)^\top = -\sum_{k=1}^n \langle \bar{\nabla}_{E_i} N, E_k \rangle E_k - \langle \bar{\nabla}_{E_i} N, \frac{\xi}{\|\xi\|} \rangle \frac{\xi}{\|\xi\|} = \frac{\lambda_i}{\lambda} E_i,$$

for $1 \leq i \leq n$. Finally, taking into account that $A^N(\frac{\xi}{\|\xi\|}) = -(\bar{\nabla}_{\frac{\xi}{\|\xi\|}} N)^\top = 0$, we get

$$\|A_{(p,t)}^N\|^2 = \sum_{i=1}^n \left(\frac{\lambda_i}{\lambda(p, t)} \right)^2 = \frac{1}{\lambda^2(p, t)} \|A_p^\eta\|^2.$$

□

Corollary 2.2. *The ϵ -truncated cone $C_\epsilon M$ is minimal in \overline{M} if, and only if, M is minimal in Ξ .*

Proof. If we let $H_{(p,t)}$ be the mean curvature vector of Φ at (p,t) , and H_p be that of φ at p , it follows from the previous result that $\|H_{(p,t)}\| = \frac{1}{\lambda}\|H_p\|$. This proves the corollary. \square

The following technical result, which is an adapted version of Theorem 4.1 of [4], will be quite useful in the proof of Proposition 2.4. In order to state it properly, we let ∇ denote the Levi-Civita connection of $C_\epsilon M$.

Lemma 2.3. *Fix $p \in M$ and, in a neighborhood Ω of p in M , an orthonormal frame (e_1, \dots, e_n) , geodesic at p . If E_1, \dots, E_n are the vector fields on $\Phi(\Omega \times (-\epsilon, 0])$, respectively obtained from e_1, \dots, e_n by parallel transport along the integral curves of $\xi/\|\xi\|$ that intersect Ω , then*

$$(5) \quad \nabla_{E_i} E_i = -\frac{\psi_\xi}{\|\xi\|^2} \xi$$

at (p, t) , for all $1 \leq i \leq n$.

Proof. Choose vector fields (η_1, \dots, η_k) on Ω , such that $(e_1, \dots, e_n, \eta_1, \dots, \eta_k)$ is an orthonormal frame adapted to the isometric immersion φ . Also, let N_1, \dots, N_k be the vector fields on $\Phi(\Omega \times (-\epsilon, 0])$, respectively obtained from η_1, \dots, η_k by parallel transport along the integral curves of $\xi/\|\xi\|$ that intersect Ω . Then, the orthonormal frame $(E_1, \dots, E_n, \frac{\xi}{\|\xi\|}, N_1, \dots, N_k)$ on $\Phi(\Omega \times (-\epsilon, 0])$ is adapted to the isometric immersion Φ .

We shall compute $\overline{\nabla}_{E_i} E_i$ at p and take its tangential component along $C_\epsilon M$. To this end, note first of all that

$$(6) \quad \langle \overline{\nabla}_{E_i} E_i, \xi \rangle = -\langle E_i, \overline{\nabla}_{E_i} \xi \rangle = -\psi_\xi.$$

As before, letting \overline{R} denote the curvature operator of \overline{M} , it follows from the parallelism of the E_i 's, together with the fact that \overline{M} has constant sectional curvature, that

$$\begin{aligned} (7) \quad \frac{d}{dt} \langle \overline{\nabla}_{E_i} E_i, E_l \rangle &= \frac{1}{\|\xi\|} \langle \overline{\nabla}_\xi \overline{\nabla}_{E_i} E_i, E_l \rangle \\ &= \frac{1}{\|\xi\|} \langle \overline{R}(\xi, E_i) E_i + \overline{\nabla}_{E_i} \overline{\nabla}_\xi E_i + \overline{\nabla}_{[\xi, E_i]} E_i, E_l \rangle \\ &= \frac{1}{\|\xi\|} (\langle \overline{R}(\xi, E_i) E_i, E_l \rangle - \langle \overline{\nabla}_{\overline{\nabla}_{E_i} \xi} E_i, E_l \rangle) \\ &= -\frac{\psi_\xi}{\|\xi\|} \langle \overline{\nabla}_{E_i} E_i, E_l \rangle, \end{aligned}$$

Also as before, let D and ∇ respectively denote the Levi-Civita connections of Ξ^{n+k} and M^n . Since (e_1, \dots, e_n) is geodesic at p (on M), we get

$$(8) \quad \langle \overline{\nabla}_{E_i} E_i, E_l \rangle_p = \langle D_{e_i} e_i, e_l \rangle_p = \langle \nabla_{e_i} e_i, e_l \rangle_p = 0.$$

Therefore, by solving Cauchy's problem formed by (7) and (8), we get

$$(9) \quad \langle \overline{\nabla}_{E_i} E_i, E_l \rangle_{(p,t)} = 0,$$

for $-\epsilon \leq t \leq 0$.

Analogously to (7), we obtain

$$(10) \quad \frac{d}{dt} \langle \overline{\nabla}_{E_i} E_i, N_\beta \rangle = -\frac{\psi_\xi}{\|\xi\|} \langle \overline{\nabla}_{E_i} E_i, N_\beta \rangle.$$

On the other hand, letting $A_\beta : T_p M \rightarrow T_p M$ denote the shape operator of φ in the direction of η_β and writing $A_\beta e_i = \sum_{j=1}^n h_{ij}^\beta e_j$, we get

$$(11) \quad \langle \bar{\nabla}_{E_i} E_i, N_\beta \rangle_p = \langle D_{e_i} e_i, \eta_\beta \rangle_p = \langle A_\beta e_i, e_i \rangle = h_{ii}^\beta.$$

Thus, by solving Cauchy's problem formed by (10) and (11), we arrive at

$$(12) \quad \langle \bar{\nabla}_{E_i} E_i, N_\beta \rangle_{(p,t)} = h_{ii}^\beta \exp \left(- \int_0^t \frac{\psi_\xi}{\|\xi\|}(s) ds \right) = \frac{h_{ii}^\beta}{\lambda(p,t)}.$$

Finally, a simple computation shows that

$$(13) \quad \langle \bar{\nabla}_{E_i} E_i, \frac{\xi}{\|\xi\|} \rangle = - \frac{\psi_\xi}{\|\xi\|}.$$

Therefore, it follows from (9), (12) and (13) that, at the point (p, t) , we have

$$(14) \quad \bar{\nabla}_{E_i} E_i = - \frac{\psi_\xi}{\|\xi\|} \frac{\xi}{\|\xi\|} + \frac{1}{\lambda(p,t)} \sum_{\beta=1}^k h_{ii}^\beta N_\beta.$$

From this equality, (5) follows promptly. \square

Given a smooth function $F \in C^\infty(C_\epsilon M)$ and $t \in [-\epsilon, 0]$, we let $F_t \in C^\infty(M)$ be the (smooth) function such that $F_t(p) = F(p, t)$, for all $p \in M$. The next result relates the Laplacians of F and F_t .

Proposition 2.4. *In the above notations, for $F \in C^\infty(C_\epsilon M)$, we have*

$$\begin{aligned} \Delta F(p, t) &= \frac{1}{\lambda^2(p, t)} \left(\Delta F_t(p) - \frac{1}{\lambda(p, t)} \langle \text{grad}(F_t), \text{grad}(\lambda_t) \rangle_p \right) \\ &\quad + n \frac{\lambda'(p, t)}{\lambda(p, t)} \frac{\partial F}{\partial t} + \frac{\partial^2 F}{\partial t^2}, \end{aligned}$$

where λ' denotes $\frac{\partial \lambda}{\partial t}$ and grad denotes gradient in M .

Proof. Fix a point $p \in M$ and, in a neighborhood $\Omega \subset M$ of p , an orthonormal frame $(e_1, \dots, e_n, \eta_1, \dots, \eta_k)$, adapted to φ , such that (e_1, \dots, e_n) is geodesic at p . As in the proof of Proposition 2.1, parallel transport this frame along the integral curves of $\xi/\|\xi\|$ to get vector fields $E_1, \dots, E_n, N_1, \dots, N_k$ along $\Phi(\Omega \times (-\epsilon, 0])$. Then, $(E_1, \dots, E_n, \frac{\xi}{\|\xi\|}, N_1, \dots, N_k)$ is an orthonormal frame adapted to the immersion Φ .

The Laplacian of F is given by

$$(15) \quad \begin{aligned} \Delta F &= \sum_{i=1}^n E_i(E_i(F)) + \frac{\xi}{\|\xi\|} \left(\frac{\xi}{\|\xi\|}(F) \right) \\ &\quad - \sum_{i=1}^n (\nabla_{E_i} E_i)(F) - (\nabla_{\xi/\|\xi\|} \xi/\|\xi\|)(F). \end{aligned}$$

It follows from Lemma 2.3 that

$$(16) \quad (\nabla_{E_i} E_i)(F) = \left(- \frac{\psi_\xi}{\|\xi\|^2} \xi \right) (F) = - \frac{\psi_\xi}{\|\xi\|} \frac{\partial F}{\partial t}.$$

Now, let us compute the summands $E_i(E_i(F))(q, t)$, where $q \in \Omega$ and $t \in [-\epsilon, 0]$. To this end, take a smooth curve $\alpha : (-\delta, \delta) \rightarrow M$, such that $\alpha(0) = q$ and $\alpha'(0) = e_i(q)$. Then, consider the parametrized surface $f : (-\delta, \delta) \times [-\epsilon, 0] \rightarrow \overline{M}$, such that

$$f(s, t) = \Psi(t, \varphi(\alpha(s))),$$

for $(s, t) \in (-\delta, \delta) \times [-\epsilon, 0]$. (Note that the image of f is contained in $C_\epsilon M$.) Lemma 3.4 of [8] gives

$$\frac{D}{dt} \frac{\partial f}{\partial s} = \frac{D}{ds} \frac{\partial f}{\partial t} = \frac{D}{ds} \frac{\xi}{\|\xi\|} = \bar{\nabla}_{\frac{\partial f}{\partial s}} \frac{\xi}{\|\xi\|} = \frac{\psi_\xi}{\|\xi\|} \frac{\partial f}{\partial s} + \frac{\partial f}{\partial s} \left(\frac{1}{\|\xi\|} \right) \xi,$$

which, in turn, implies

$$\frac{d}{dt} \left\langle \frac{\partial f}{\partial s}, E_j \right\rangle = \frac{\psi_\xi}{\|\xi\|} \left\langle \frac{\partial f}{\partial s}, E_j \right\rangle.$$

Since $\left\langle \frac{\partial f}{\partial s}, E_j \right\rangle_{(q,0)} = \langle E_i, E_j \rangle_{(q,0)} = \langle e_i(q), e_j(q) \rangle = \delta_{ij}$, in solving the Cauchy problem for $\left\langle \frac{\partial f}{\partial s}, E_j \right\rangle$ so obtained, we get

$$\left\langle \frac{\partial f}{\partial s}, E_i \right\rangle_{(q,t)} = \exp \left(\int_0^t \frac{\psi_\xi}{\|\xi\|} (q, u) du \right) = \lambda(q, t)$$

and, for $j \neq i$,

$$\left\langle \frac{\partial f}{\partial s}, E_j \right\rangle_{(q,t)} = 0.$$

Moreover, direct computation shows that $\frac{d}{dt} \left\langle \frac{\partial f}{\partial s}, \xi \right\rangle = \frac{\psi_\xi}{\|\xi\|} \left\langle \frac{\partial f}{\partial s}, \xi \right\rangle$; but, since $\left\langle \frac{\partial f}{\partial s}, \xi \right\rangle_{(q,0)} = \langle E_i, \xi \rangle_{(q,0)} = 0$, it follows from the uniqueness of the solution of a Cauchy problem that $\left\langle \frac{\partial f}{\partial s}, \xi \right\rangle_{(q,t)} = 0$, for $t \in [-\epsilon, 0]$.

Since $\frac{\partial f}{\partial s}$ is tangent to the cone, the previous computations show that, at the point (q, t) ,

$$(17) \quad \frac{\partial f}{\partial s} = \sum_{j=1}^n \left\langle \frac{\partial f}{\partial s}, E_j \right\rangle E_j + \left\langle \frac{\partial f}{\partial s}, \frac{\xi}{\|\xi\|} \right\rangle \frac{\xi}{\|\xi\|} = \left\langle \frac{\partial f}{\partial s}, E_i \right\rangle E_i = \lambda E_i.$$

Therefore,

$$E_i(F)(q, t) = \frac{1}{\lambda(q, t)} \frac{\partial f}{\partial s}(q, t)(F) = \frac{1}{\lambda(q, t)} dF_t(e_i(q)) = \frac{1}{\lambda(q, t)} \langle \text{grad}(F_t), e_i \rangle_q,$$

for all points $(q, t) \in \Omega \times [-\epsilon, 0]$ and all $F \in C^\infty(C_\epsilon M)$. Thus,

$$E_i(E_i(F))(q, t) = \frac{1}{\lambda(q, t)} \langle \text{grad}((E_i(F))_t), e_i \rangle_q.$$

On the other hand, at the point q we get

$$\begin{aligned} \text{grad}((E_i(F))_t) &= \frac{1}{\lambda_t} \text{grad} \langle \text{grad}(F_t), e_i \rangle + \langle \text{grad}(F_t), e_i \rangle \text{grad} \left(\frac{1}{\lambda_t} \right) \\ &= \frac{1}{\lambda_t} \text{grad}(e_i(F_t)) - \langle \text{grad}(F_t), e_i \rangle \frac{1}{\lambda_t^2} \text{grad}(\lambda_t) \end{aligned}$$

and, hence,

$$\begin{aligned} E_i(E_i(F))(q, t) &= \frac{1}{\lambda(q, t)} \left(\frac{1}{\lambda(q, t)} e_i(e_i(F_t))(q) - \frac{\langle \text{grad}(F_t), e_i \rangle_q}{\lambda^2(q, t)} e_i(q)(\lambda_t) \right) \\ &= \frac{1}{\lambda^2(q, t)} \left(e_i(e_i(F_t))(q) - \frac{\langle \text{grad}(F_t), e_i \rangle_q}{\lambda(q, t)} \langle \text{grad}(\lambda_t), e_i \rangle_q \right), \end{aligned}$$

at all points $(q, t) \in \Omega \times [-\epsilon, 0]$.

By using the fact that the frame (e_1, \dots, e_n) is geodesic at the point p we get, at the point (p, t) ,

$$(18) \quad \sum_{i=1}^n E_i(E_i(F)) = \frac{1}{\lambda^2} \left(\Delta F_t - \frac{1}{\lambda} \langle \text{grad}(F_t), \text{grad}(\lambda_t) \rangle \right).$$

If we let $(\cdot)^\top$ denote orthogonal projection on $T(C_\epsilon M)$, we compute

$$\begin{aligned}\nabla_{\xi/\|\xi\|}\xi/\|\xi\| &= \frac{1}{\|\xi\|}\nabla_\xi\xi/\|\xi\| = \frac{1}{\|\xi\|}\left(\frac{1}{\|\xi\|}\nabla_\xi\xi + \xi\left(\frac{1}{\|\xi\|}\right)\xi\right) \\ &= \frac{1}{\|\xi\|}\left(\frac{1}{\|\xi\|}(\tilde{\nabla}_\xi\xi)^\top - \frac{\|\xi\|\psi_\xi}{\|\xi\|^2}\xi\right) \\ &= \frac{1}{\|\xi\|}\left(\frac{1}{\|\xi\|}(\psi_\xi\xi)^\top - \frac{\psi_\xi}{\|\xi\|}\xi\right) = 0;\end{aligned}$$

hence, $(\nabla_{\xi/\|\xi\|}\xi/\|\xi\|)(F) = 0$.

Substituting this last computation in (15), and taking (16) and (18) into account, we finally arrive at

$$\begin{aligned}\Delta F(p, t) &= \frac{1}{\lambda^2(p, t)}\left(\Delta F_t(p) - \frac{1}{\lambda(p, t)}\langle \text{grad}(F_t), \text{grad}(\lambda_t) \rangle_p\right) \\ &\quad + n\frac{\psi_\xi}{\|\xi\|}\frac{\partial F}{\partial t}(p, t) + \frac{\partial^2 F}{\partial t^2}(p, t),\end{aligned}$$

and a simple computation shows that $\frac{\psi_\xi}{\|\xi\|} = \frac{\lambda'}{\lambda}$. \square

3. ON THE UNSTABILITY OF MINIMAL CONES

By Corollary 2.2, we know that M^n is minimal in Ξ^{n+k} if, and only if, $C_\epsilon M$ is minimal in \overline{M}_c^{n+k+1} . Since minimal immersions are precisely the critical points of the area functional with respect to variations that fix the boundary, for a given M , minimal in Ξ^{n+k} , it makes sense to consider the problem of stability of $C_\epsilon M$ with respect to normal variations that fix its boundary. In this section, we address this problem in the case in which $k = 1$, i.e., when M^n is a hypersurface of Ξ^{n+1} . This will extend the analysis made in [10], where $\overline{M} = R^{n+2}$, $\Xi = S^{n+1}$ and $\xi(x) = x$.

Throughout the rest of this paper, until further notice, we stick to the notations of the previous section. In particular, \overline{M} continues to be of constant sectional curvature, equal to c ; also, whenever we let η denote a unit vector field normal to M in Ξ , we shall let N denote the unit vector field normal to $C_\epsilon M$ in \overline{M} , obtained by parallel transport of η along the integral curves of $\frac{\xi}{\|\xi\|}$ that intersect M . We start with the following auxiliary result.

Lemma 3.1. *Let Ξ^{n+1} be oriented by the unit normal vector field $\frac{\xi}{\|\xi\|}$, and let M^n be a minimal hypersurface of Ξ^{n+1} , oriented by the unit vector field $\eta \in \mathfrak{X}(M)^\perp \cap \mathfrak{X}(\Xi)$. If $C_\epsilon M$ is oriented by N , then its volume element is given by $\lambda^n dM \wedge dt$, where dM stands for the volume element of M .*

Proof. Let (e_1, \dots, e_n) be a positive orthonormal frame, defined in an open set $\Omega \subset M$. If $(\theta_1, \dots, \theta_n)$ denotes the corresponding coframe, then $dM = \theta_1 \wedge \dots \wedge \theta_n$ in Ω .

Let E_1, \dots, E_n be the vector fields on $\Phi(\Omega \times (-\epsilon, \epsilon))$ obtained from the e_i 's by parallel transport along the integral curves of $\xi/\|\xi\|$ that intersect Ω . For $p \in \Omega$, the orthonormal basis (e_1, \dots, e_n, η) of $T_p\Xi$ is positively oriented; hence, the orthonormal basis $(e_1, \dots, e_n, \eta, -\frac{\xi}{\|\xi\|})$ of $T_p\overline{M}$ is also positively oriented. It follows that the orthonormal basis $(E_1, \dots, E_n, N, -\frac{\xi}{\|\xi\|})_{(p,t)}$ of $T_{(p,t)}\overline{M}$ is positively oriented and, thus, $(E_1, \dots, E_n, \frac{\xi}{\|\xi\|}, N)_{(p,t)}$ is also a positively oriented orthonormal basis of $T_{(p,t)}\overline{M}$, for all $(p, t) \in \Omega \times (-\epsilon, \epsilon)$. Therefore, $(E_1, \dots, E_n, \frac{\xi}{\|\xi\|})$ is a positively oriented orthonormal basis of $T_{(p,t)}(C_\epsilon M)$.

Now, let $\alpha_i : (-\delta, \delta) \rightarrow M$ be a smooth curve such that $\alpha_i(0) = p$ and $\alpha'_i(0) = e_i(p)$; if $f_i : (-\delta, \delta) \times (-\epsilon, 0] \rightarrow \overline{M}$ is the parametrized surface such that $f_i(s, t) =$

$\Psi(t, \varphi(\alpha_i(s)))$, we show in (17) that

$$E_i(p, t) = \frac{1}{\lambda(p, t)} \frac{\partial f_i}{\partial s}(0, t).$$

By the canonical identification of $T_{(p,t)}(M \times (-\epsilon, 0])$ and $T_p M \oplus \mathbb{R}$, we have

$$\Phi_*(e_i(p) \oplus 0)_{(p,t)} = \frac{d}{ds} \Big|_{s=0} \Phi(\alpha_i(s), t) = \frac{d}{ds} \Big|_{s=0} f_i(s, t) = \frac{\partial f_i}{\partial s}(0, t)$$

and, thus,

$$\Phi_* \left(\frac{e_i(p)}{\lambda(p, t)} \oplus 0 \right)_{(p,t)} = E_i(p, t).$$

Therefore, by using the canonical identification of $T_{\Phi(p,t)}(C_\epsilon M)$ and $\Phi_*(T_{(p,t)}(M \times (-\epsilon, 0]))$, we get

$$\lambda^n(dM \wedge dt)(E_1, \dots, E_n, \frac{\xi}{\|\xi\|}) = \lambda^n(dM \wedge dt)(\frac{e_1}{\lambda} \oplus 0, \dots, \frac{e_n}{\lambda} \oplus 0, 0 \oplus \partial t) = 1,$$

which concludes the proof. \square

Given a minimal isometric immersion $\varphi : M^n \rightarrow \Xi^{n+1}$, the following proposition computes the second variation of area for the corresponding ϵ -truncated cone $C_\epsilon M$. As usual, for $F \in C^\infty(C_\epsilon M)$, we let $I(F)$ denote the index form of $C_\epsilon M$ in the direction of $V = FN$.

Proposition 3.2. *Let M^n be a closed, oriented, minimal hypersurface of Ξ^{n+1} . Suppose that the function $\lambda(p, t)$ does not depend on the point p , and let $N(p, t)$ denote the unit normal vector field that orients $C_\epsilon M$. If $F \in C^\infty(C_\epsilon M)$ is such that $F(p, -\epsilon) = F(p, 0) = 0$, for each $p \in M$, then*

$$\begin{aligned} I(F) = \int_{M \times [-\epsilon, 0]} & F \lambda^{n-2} \left(-\Delta F_t - n \lambda \lambda' \frac{\partial F}{\partial t} - \lambda^2 \frac{\partial^2 F}{\partial t^2} \right. \\ & \left. - c(n+1) \lambda^2 F - \|A^\eta\|^2 F \right) dM \wedge dt. \end{aligned}$$

Proof. It is a classical fact (cf. [2], [10] or [11]) that

$$I(F) = \int_{C_\epsilon M} (-F \Delta F - (\bar{R} + \|A^N\|^2) F^2) d(C_\epsilon M),$$

where $\bar{R} = \overline{\text{Ric}}(N, N)$, and $\overline{\text{Ric}}$ denotes the Ricci tensor of \bar{M} . Therefore, it follows from the formulae of propositions 2.1 and 2.4, together with the fact that \bar{M} has sectional curvature constant and equal to c and $\lambda(p, t)$ does not depend on p , that the integrand of the right hand side equals

$$\begin{aligned} & -F \left(\frac{1}{\lambda^2} \Delta F_t + n \frac{\lambda'}{\lambda} \frac{\partial F}{\partial t} + \frac{\partial^2 F}{\partial t^2} \right) - c(n+1) F^2 - \|A^\eta\|^2 \frac{F^2}{\lambda^2} = \\ & = \frac{F}{\lambda^2} \left(-\Delta F_t - n \lambda \lambda' \frac{\partial F}{\partial t} - \lambda^2 \frac{\partial^2 F}{\partial t^2} - c(n+1) \lambda^2 F - \|A^\eta\|^2 F \right). \end{aligned}$$

Finally, it now suffices to apply the result of the previous lemma and integrate on $M \times [-\epsilon, 0]$. \square

Now, let $C_0^\infty[-\epsilon, 0] = \{g \in C^\infty[-\epsilon, 0]; g(-\epsilon) = g(0) = 0\}$. Following [10], the previous proposition motivates the introduction of the linear differential operators $\mathcal{L}_1 : C^\infty(M) \rightarrow C^\infty(M)$ and $\mathcal{L}_2 : C_0^\infty[-\epsilon, 0] \rightarrow C^\infty[-\epsilon, 0]$, given by

$$\mathcal{L}_1(f) = -\Delta f - \|A^\eta\|^2 f \quad \text{and} \quad \mathcal{L}_2(g) = -\lambda^2 g'' - n \lambda \lambda' g' - c(n+1) \lambda^2 g.$$

Standard elliptic theory (cf. [7]) shows that \mathcal{L}_1 can be diagonalized by a sequence $(f_i)_{i \geq 1}$ of smooth eigenfunctions, orthogonal in $L^2(M)$ and whose sequence $(\lambda_i)_{i \geq 1}$

of corresponding eigenvalues satisfy $\lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty$; moreover, each $f \in C^\infty(M)$ can be uniquely written as $f = \sum_{i \geq 1} a_i f_i$, for some $a_i \in \mathbb{R}$.

On the other hand, equation $\mathcal{L}_2(g) = \delta g$, for $\delta \in \mathbb{R}$, is equivalent to

$$-\lambda^2 g'' - n\lambda\lambda' g' - c(n+1)\lambda^2 g - \delta g = 0,$$

or (after multiplying both sides by $-\lambda^{n-2}$) yet to

$$(19) \quad (\lambda^n g')' + c(n+1)\lambda^n g + \delta\lambda^{n-2}g = 0.$$

Hence, the elementary theory of regular Sturm-Liouville problems (cf. [6]) shows that \mathcal{L}_2 can also be diagonalized by a sequence $(g_i)_{i \geq 1}$ of smooth eigenfunctions, orthogonal in $L_w^2[-\epsilon, 0]$ with respect to the weight $w = \lambda^{n-2}$ and whose sequence $(\delta_i)_{i \geq 1}$ of corresponding eigenvalues satisfy $\delta_1 \leq \delta_2 \leq \dots \rightarrow +\infty$; moreover, each $g \in C_0^\infty[-\epsilon, 0]$ can be uniquely written as $g = \sum_{i \geq 1} a_i g_i$, for some $a_i \in \mathbb{R}$.

In view of all of the above, the proof of the following result parallels that of Lemma 6.1.6 of [10]. For the sake of completeness, we present it here.

Theorem 3.3. *With notations as in Proposition 3.2, it is possible to choose F such that $I(F) < 0$ if, and only if, $\lambda_1 + \delta_1 < 0$, where λ_1 and δ_1 stand, respectively, to the first eigenvalues of \mathcal{L}_1 and \mathcal{L}_2 .*

Proof. For a fixed $p \in M$, we have $F(p, \cdot) \in C_0^\infty[-\epsilon, 0]$. Therefore, the discussion on the diagonalization of \mathcal{L}_2 gives $F(p, t) = \sum_{j \geq 1} a_j(p) g_j(t)$, for some $a_j \in C^\infty(M)$; hence, by invoking the discussion on the diagonalization of \mathcal{L}_1 , we get

$$F(p, t) = \sum_{i, j \geq 1} a_{ij} f_i(p) g_j(t),$$

for some $a_{ij} \in \mathbb{R}$.

It now follows from the result of Proposition 3.2 that

$$\begin{aligned} I(F) &= \int_{M \times [-\epsilon, 0]} \lambda^{n-2} \sum_{i, j \geq 1} a_{ij} f_i g_j \sum_{k, l \geq 1} (a_{kl} \mathcal{L}_1(f_k) g_l + a_{kl} f_k \mathcal{L}_2(g_l)) dM \wedge dt \\ &= \int_{M \times [-\epsilon, 0]} \lambda^{n-2} \sum_{i, j \geq 1} a_{ij} f_i g_j \sum_{k, l \geq 1} a_{kl} (\lambda_k + \delta_l) f_k g_l dM \wedge dt \\ &= \sum_{i, j, k, l \geq 1} a_{ij} a_{kl} (\lambda_k + \delta_l) \int_{M \times [-\epsilon, 0]} f_i f_k g_j g_l \lambda^{n-2} dM \wedge dt. \end{aligned}$$

From here, the orthogonality conditions on the eigenfunctions of \mathcal{L}_1 and \mathcal{L}_2 easily give

$$I(F) = \sum_{i, j \geq 1} a_{ij}^2 (\lambda_i + \delta_j) \left(\int_M f_i^2 dM \right) \left(\int_{-\epsilon}^0 g_j^2 \lambda^{n-2} dt \right).$$

Therefore, if $I(F) < 0$, then some factor $\lambda_i + \delta_j$ is negative and, hence, $\lambda_1 + \delta_1 < 0$ (since $\lambda_1 \leq \lambda_i$ and $\delta_1 \leq \delta_j$); conversely, if $\lambda_1 + \delta_1 < 0$, choose $F(p, t) = f_1(p) g_1(t)$ to get $I(F) < 0$. \square

For future reference, we recall the standard variational characterization of λ_1 (cf. [5] or [7]): for a given $f \in C^\infty(M) \setminus \{0\}$, let the Rayleigh quotient of f with respect to \mathcal{L}_1 be defined by

$$(20) \quad RQ[f] = \frac{\int_M -f(\Delta f + \|A^\eta\|^2 f) dM}{\int_M f^2 dM};$$

Then,

$$(21) \quad \lambda_1 = \min\{RQ[f]; f \in C^\infty(M) \setminus \{0\}\},$$

with equality if, and only if, f is an eigenfunction of \mathcal{L}_1 with respect to λ_1 .

In what concerns δ_1 , given $g \in C_0^\infty[-\epsilon, 0] \setminus \{0\}$, let the Rayleigh quotient of g with respect to (19) be defined by

$$(22) \quad RQ[g] = \frac{\int_{-\epsilon}^0 \lambda^n ((g')^2 - c(n+1)g^2) dt}{\int_{-\epsilon}^0 \lambda^{n-2} g^2 dt}.$$

Then (cf. [6]),

$$(23) \quad \delta_1 = \min\{RQ[g]; g \in C_0^\infty[-\epsilon, 0] \setminus \{0\}\},$$

with equality if, and only if, g is an eigenfunction of \mathcal{L}_2 with respect to δ_1 .

4. MINIMAL CONES IN WARPED PRODUCTS

Let B and F be Riemannian manifolds and $f : B \rightarrow \mathbb{R}$ be a smooth positive function. The *warped product* $M = B \times_f F$ is the product manifold $B \times F$, furnished with the Riemannian metric

$$g = \pi_B^*(g_B) + (f \circ \pi_B)^2 \pi_F^*(g_F),$$

where π_B and π_F denote the canonical projections from $B \times F$ onto B and F and g_B and g_F denote the Riemannian metrics of B and F , respectively.

In this section, we shall consider a warped product $\overline{M}_c^{n+2} = I \times_f F^{n+1}$, with $I \subset \mathbb{R}$, $f(0) = 1$ and having constant sectional curvature, equal to c . By Proposition 7.42 of [9], this last condition amounts to the fact that F^{n+1} should have constant sectional curvature k , such that

$$\frac{f''}{f} = -c = \frac{(f')^2 - k}{f^2}$$

on I .

In what concerns our previous discussion of cones, we get the following consequence of Proposition 2.4 when $\overline{M} = I \times_f F$, a warped product for which $I \subset \mathbb{R}$.

Corollary 4.1. *Let $\overline{M}_c^{n+2} = I \times_f F^{n+1}$, with $f(0) = 1$. If M^n is a closed Riemannian manifold and $\varphi : M^n \rightarrow F^{n+1}$ is an isometric immersion, then*

$$\Delta L(t, p) = \frac{1}{f^2(t)} \Delta L_t(p) + n \frac{f'(t)}{f(t)} \frac{\partial L}{\partial t} + \frac{\partial^2 L}{\partial t^2},$$

for all $L \in C^\infty(I \times_f M^n)$.

Proof. It is a standard fact (cf. [9]) that, in $I \times_f F^{n+1}$, the vector field $\xi = (f \circ \pi_I) \partial_t$ is closed and conformal, with conformal factor $\psi_\xi = f' \circ \pi_I$. Moreover, $\xi \neq 0$, since f is positive. The flux Ψ of $\frac{\xi}{\|\xi\|} = \partial_t$ is given by

$$\Psi(t, (t_0, p)) = (t + t_0, p),$$

and it is clear that the submanifolds $\{t_0\} \times F^{n+1}$, with $t_0 \in I$, are leaves of ξ^\perp .

Now, let $\varphi : M^n \rightarrow F^{n+1}$ be an isometric immersion from a closed Riemannian manifold M^n into F^{n+1} . Since $f(0) = 1$, the leaf $\{0\} \times F^{n+1}$ of ξ^\perp (with the metric induced from $I \times_f F^{n+1}$) is isometric to F^{n+1} ; therefore, we can (and do) assume that φ takes M into $\{0\} \times F^{n+1}$. The compactness of M guarantees the existence of $\epsilon > 0$ such that the ϵ -truncated cone $C_\epsilon M$ is given by the immersion

$$\Phi(p, t) = \Psi(t, (0, \varphi(p))) = (t, \varphi(p)),$$

for $t \in [-\epsilon, 0]$ and $p \in M^n$. (Actually, Φ continues to be an immersion even if we change $t \in [-\epsilon, 0]$ by $t \in I$.) Moreover, $C_\epsilon M$ is isometric to the warped product $[-\epsilon, 0] \times_f M^n$.

In view of the above, the function λ of (2) is such that

$$\lambda(p, s) = \exp \left(\int_0^s \frac{\psi_\xi}{\|\xi\|} (\Psi(t, \varphi(p))) dt \right) = \exp \left(\int_0^s \frac{f'}{f}(t) dt \right) = f(s).$$

In particular, $\lambda_s : M^n \rightarrow \mathbb{R}$ is constant, for all $s \in [-\epsilon, 0]$, and it suffices to apply the result of Proposition 2.4. \square

From now on, let M^n be a closed, minimal and non totally geodesic hypersurface of $F^{n+1} \approx \{0\} \times F^{n+1}$. According to the proof of the previous corollary, we shall identify the ϵ -truncated cone $C_\epsilon M$ with the warped product $[-\epsilon, 0] \times_f M^n$, canonically immersed into \overline{M}_c^{n+2} .

If (as before) $N(t, p)$ stands for the unit normal vector field of $C_\epsilon M$ and $G \in C^\infty(C_\epsilon M)$ is such that $G(-\epsilon, p) = G(0, p) = 0$ for each $p \in M$, then Proposition 3.2 gives

$$(24) \quad I(G) = \int_{M \times [-\epsilon, 0]} G f^{n-2} \left(-\Delta G_t - n f f' \frac{\partial G}{\partial t} - f^2 \frac{\partial^2 G}{\partial t^2} - c(n+1) f^2 G - \|A\|^2 G \right) dM \wedge dt,$$

where $\|A\|$ stands for the norm of the second fundamental form of the immersion of M^n into F^{n+1} and Δ for the Laplacian operator of M^n .

In this case, the linear differential operators $\mathcal{L}_1 : C^\infty(M) \rightarrow C^\infty(M)$ and $\mathcal{L}_2 : C_0^\infty[-\epsilon, 0] \rightarrow C^\infty[-\epsilon, 0]$ are given by

$$(25) \quad \mathcal{L}_1(g) = -\Delta g - \|A\|^2 g,$$

for $g \in C^\infty(M)$, and

$$(26) \quad \mathcal{L}_2(h) = -f^2 h'' - n f f' h' - c(n+1) f^2 h,$$

for $h \in C_0^\infty[-\epsilon, 0]$.

We want to apply Theorem 3.3 to the case in which \overline{M}_c^{n+2} is the Euclidean sphere \mathbb{S}^{n+2} . To this end, let $I = (-\frac{\pi}{2}, \frac{\pi}{2})$, $f(t) = \cos t$, $F^{n+1} = \mathbb{S}^{n+1}$, $N = (0, \dots, 0, 1) \in \mathbb{S}^{n+2}$ and consider \mathbb{S}^{n+1} as the equator of \mathbb{S}^{n+2} which has N as North pole; also, identify $x = (x_1, \dots, x_{n+2}) \in \mathbb{S}^{n+1}$ to the point $x = (x_1, \dots, x_{n+2}, 0) \in \mathbb{S}^{n+2}$. With these conventions, the map

$$(t, x) \mapsto (\cos t)x + (\sin t)N$$

defines an isometry between $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{S}^{n+1}$ and $\mathbb{S}^{n+2} \setminus \{\pm N\}$.

Once again, let $\varphi : M^n \rightarrow \mathbb{S}^{n+1}$ be a closed, minimal, non totally geodesic hypersurface of \mathbb{S}^{n+1} . The ϵ -truncated cone $C_\epsilon M$ can be seen as the image of the isometric immersion

$$(27) \quad \begin{aligned} \Phi : [-\epsilon, 0] \times M^n &\longrightarrow \mathbb{S}^{n+2} \\ (t, x) &\longmapsto (\cos t)x + (\sin t)N. \end{aligned}$$

In order to get an upper estimate for λ_1 , recall from (20) and (21) that

$$(28) \quad \lambda_1 \leq \frac{\int_M -g(\Delta g + \|A\|^2 g) dM}{\int_M g^2 dM},$$

for any $g \in C^\infty(M) \setminus \{0\}$. Following [10], let $\tau > 0$ and $g_\tau = (\|A\|^2 + \tau)^{1/2}$. Simons' formula for $\Delta(\|A\|^2)$ (cf. [3] or [10] – recall that F is also of constant sectional curvature) easily gives

$$g_\tau \Delta g_\tau \geq n\|A\|^2 - \|A\|^4.$$

Hence, by taking g_τ in place of g in (28), we arrive at

$$\lambda_1 \leq -\frac{\int_M (n + \tau) \|A\|^2 dM}{\int_M (\|A\|^2 + \tau) dM},$$

By letting $\tau \rightarrow 0$, and taking into account that $\int_M \|A\|^2 dM > 0$ (since M is not totally geodesic), we get $\lambda_1 \leq -n$.

In what concerns δ_1 , equation (26) gives

$$\mathcal{L}_2(h) = -(\cos^2 t)h'' + n(\sin t \cos t)h' - (n+1)(\cos^2 t)h,$$

so that (arguing as in the discussion that precedes the statement of Theorem 3.3) $\mathcal{L}_2(h) = \delta h$ is equivalent to

$$(29) \quad ((\cos^n t)h')' + (n+1)(\cos^n t)h + \delta(\cos^{n-2} t)h = 0.$$

It now follows from (22) and (23) that

$$\delta_1 \leq \frac{\int_{-\epsilon}^0 (\cos^n t)((h')^2 - (n+1)h^2)dt}{\int_{-\epsilon}^0 (\cos^{n-2} t)h^2 dt},$$

for every $h \in C_0^\infty[-\epsilon, 0] \setminus \{0\}$.

By taking

$$h(t) = \frac{\sin(\frac{\pi t}{\epsilon})}{\sqrt{\cos^{n-2} t}}$$

(which satisfies the boundary conditions), direct computations show that $h(t)^2 \cos^{n-2} t = \sin^2(\frac{\pi t}{\epsilon})$,

$$\begin{aligned} (\cos^n t)h'(t)^2 &= \frac{\pi^2}{\epsilon^2} \cos^2\left(\frac{\pi t}{\epsilon}\right) \cos^2 t + \frac{(n-2)^2}{4} \sin^2\left(\frac{\pi t}{\epsilon}\right) \sin^2 t \\ &\quad + \frac{n-2}{4} \sin\left(\frac{2\pi t}{\epsilon}\right) \sin(2t), \end{aligned}$$

and

$$(n+1)(\cos^n t)h(t)^2 = (n+1)(\cos^2 t) \sin^2\left(\frac{\pi t}{\epsilon}\right).$$

Therefore,

$$\delta_1 \leq \frac{I_1 - I_2}{I_3},$$

where

$$\begin{aligned} I_1 &= \frac{\pi^2}{\epsilon^2} \int_{-\epsilon}^0 \cos^2\left(\frac{\pi t}{\epsilon}\right) \cos^2 t dt + \frac{(n-2)^2}{4} \int_{-\epsilon}^0 \sin^2\left(\frac{\pi t}{\epsilon}\right) \sin^2 t dt \\ &\quad + \frac{n-2}{4} \int_{-\epsilon}^0 \sin\left(\frac{2\pi t}{\epsilon}\right) \sin(2t) dt, \\ I_2 &= (n+1) \int_{-\epsilon}^0 (\cos^2 t) \sin^2\left(\frac{\pi t}{\epsilon}\right) dt \quad \text{and} \quad I_3 = \int_{-\epsilon}^0 \sin^2\left(\frac{\pi t}{\epsilon}\right) dt. \end{aligned}$$

Finally, we observe that $\lim_{\epsilon \rightarrow \frac{\pi}{2}} I_1 = \frac{\pi}{2} \left(1 + \left(\frac{n-2}{4}\right)^2\right)$, $\lim_{\epsilon \rightarrow \frac{\pi}{2}} I_2 = (n+1)\frac{\pi}{8}$ and $\lim_{\epsilon \rightarrow \frac{\pi}{2}} I_3 = \frac{\pi}{4}$, so that, for $\epsilon > 0$ sufficiently close to $\frac{\pi}{2}$, we have

$$\lambda_1 + \delta_1 \leq \frac{n^2}{8} - 2n + 2.$$

Since this quadratic polynomial is negative for $2 \leq n \leq 14$, we have proved the following result.

Theorem 4.2. *Let M^n be a closed, oriented minimal hypersurface of \mathbb{S}^{n+1} . If $2 \leq n \leq 14$ and M^n is not totally geodesic, then CM is a minimal unstable hypersurface of \mathbb{S}^{n+2} .*

REFERENCES

- [1] F. J. Almgren, Jr. *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*. Ann. of Math. 85 (1966), 277-292.
- [2] J. L. M. Barbosa, M. do Carmo and J. Eschenburg. *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*. Math. Z. 197 (1988), 123-138.
- [3] A. Caminha. *On hypersurfaces into Riemannian spaces of constant sectional curvature*. Kodai Math. J. 29 (2006), 185-210.
- [4] A. Caminha. *The geometry of closed conformal vector fields on Riemannian spaces*. Bull. Braz. Math. Soc. 42 (2011), 277-300.
- [5] I. Chavel. *Eigenvalues in Riemannian Geometry*. Academic Press, London, 1984.
- [6] R. Courant and D. Hilbert. *Methods of Mathematical Physics I*. John Wiley & Sons, New York, 1989.
- [7] D. Gilbarg e N. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 1998.
- [8] M. do Carmo. *Riemannian Geometry*. Birkhäuser, Boston, 1992.
- [9] B. O'Neill. *Semi-Riemannian Geometry with Applications to Relativity*, London, Academic Press (1983).
- [10] J. Simons. *Minimal Varieties in Riemannian Manifolds*. The Annals of Mathematics (1968), 62-105.
- [11] Y. Xin. *Minimal Submanifolds and Related Topics*. Nankai Tracts in Mathematics, World Scientific, Cingapura, 2003.

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